Picard–Fuchs Equations of Dimensionally Regulated Feynman Integrals

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Regulators V

based on with Spencer Bloch, Leonardo de la Cruz, Charles Doran, Andrew Harder, Matt Kerr, Pierre Lairez, Eric Pichon-Pharabod

Any Feynman integral has the parametric representation

$$I_\Gamma(\underline{s},\underline{m};\underline{\nu},D) = \int_{\Delta_n} \Omega_\Gamma; \quad \Omega_\Gamma := \frac{\mathcal{U}_\Gamma(\underline{x})^{\nu-\frac{(L+1)D}{2}}}{\mathcal{F}_\Gamma(\underline{x})^{\nu-\frac{LD}{2}}} \prod_{i=1}^{n} x_i^{\nu_i-1}\,\Omega_0$$

with $\nu = \sum_{i=1}^{n} \nu_i$, $L \in \mathbb{N}$, $(D,\nu_1,\dots,\nu_n) \in \mathbb{C}^{n+1}$

The domain of integration is the positive quadrant

$$\Delta_n := \{x_1 \geqslant 0,\dots,x_n \geqslant 0 | [x_1,\dots,x_n] \in \mathbb{P}^{n-1}\}$$

Ω_0 is the volume form on $\mathbb{P}^{n-1}$

$$\Omega_0 = \sum_{i=1}^{n} (-1)^{i-1} x^i dx^1 \wedge \cdots \widehat{dx^i} \cdots \wedge dx^n$$

The graph polynomial is homogeneous degree $L+1$ in $\mathbb{P}^{n-1}$

$$\mathcal{F}_\Gamma(\underline{x}) = \mathcal{U}_\Gamma(\underline{x}) \times L(\underline{m}^2;\underline{x}) - \mathcal{V}_\Gamma(\underline{s},\underline{x})$$

Feynman graph polynomials

The graph polynomial is homogeneous degree $L+1$ in $\mathbb{P}^{n-1}$

$$\mathcal{F}_\Gamma(\underline{x}) = \mathcal{U}_\Gamma(\underline{x}) \times L(\underline{m}^2;\underline{x}) - \mathcal{V}_\Gamma(\underline{s},\underline{x})$$

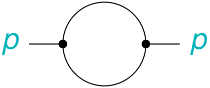
- Homogeneous polynomial of degree L with $u_{a_1,\dots,a_n} \in \{0,1\}$

$$\mathcal{U}_\Gamma(\underline{x}) = \sum_{\substack{a_1+\cdots+a_n=L\\ 0\leqslant a_i\leqslant 1}} u_{a_1,\dots,a_n} \prod_{i=1}^n x_i^{a_i}$$

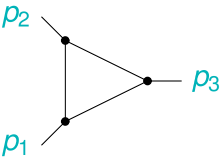
- the mass hyperplane $L(\underline{m}^2;\underline{x}) := \sum_{i=1}^n m_i^2 x_i$
- Homogeneous polynomial of degree $L+1$ with $s_{a_i,\dots,a_n}$ are linear combination of the product of the external momenta $\underline{s} = \{p_i \cdot p_j\}$

$$\mathcal{V}_\Gamma(\underline{x}) = \sum_{\substack{a_1+\cdots+a_n=L+1\\ 0\leqslant a_i\leqslant 1}} s_{a_i,\dots,a_n} \prod_{i=1}^n x_i^{a_i}$$

Feynman graph polynomials : examples

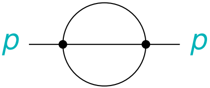


$$\begin{cases} \mathcal{U}_\circ = x_1 + x_2 \\ \mathcal{F}_\circ = \mathcal{U}_\circ \left(\sum_{i=1}^2 m_i^2 x_i\right) - p^2 x_1 x_2 \end{cases}$$



$$\begin{cases} \mathcal{U}_\triangleright = x_1 + x_2 + x_3 \\ \mathcal{V}_\triangleright = p_1^2 x_2 x_3 + p_2^2 x_1 x_3 + p_3^2 x_1 x_2 \\ \mathcal{F}_\triangleright = U_\triangleright \left(\sum_{i=1}^3 m_i^2 x_i\right) - \mathcal{V}_\triangleright \end{cases}$$

with $p_1 + p_2 + p_3 = 0$



$$\begin{cases} \mathcal{U}_\ominus = x_1 x_2 + x_1 x_3 + x_2 x_3 \\ \mathcal{F}_\ominus = U_\ominus \sum_{i=1}^3 m_i^2 x_i - p^2 x_1 x_2 x_3 \end{cases}$$

Noboru Nakanishi
Graph theory and Feynman integrals
New York : Gordon and Breach, (1971)

Feynman Integrals: singularities

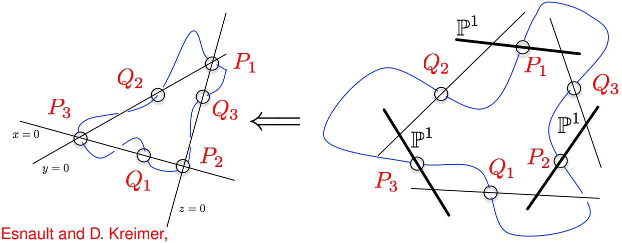
$$I_\Gamma(\underline{s},\underline{m};\underline{\nu},D)=\int_{\Delta_n}\Omega_\Gamma,\quad \Omega_\Gamma=\frac{\mathcal{U}_\Gamma(\underline{x})^{\nu-\frac{(L+1)D}{2}}}{\mathcal{F}_\Gamma(\underline{x})^{\nu-\frac{LD}{2}}}\prod_{i=1}^n x_i^{\nu_i-1}\,\Omega_0$$

- Meromorphic functions of $(D,\nu_1,\ldots,\nu_n)$ with singularities located on hyperplane defined by $\sum_{i=1}^n a_i\nu_i+a_0D=0$ with $(a_0,a_1,\ldots,a_n)\in\mathbb{Z}^{n+1}$
- All the singularities of the Feynman integrals are located on the graph hypersurface
- Generically the graph hypersurface has non-isolated singularities
- The integrand is a differential form defined in the complement of its singular locus Z_Γ in $\mathbb{P}^{n-1}$
- This is a multivalued higher transcendantal function of its physical parameters $(\underline{s},\underline{m})$

The Feynman integral *are* periods of the mixed Hodge structure after performing the appropriate blow-ups

$$\mathfrak{M}_\Gamma := H^\bullet(\widetilde{\mathbb{P}^{n-1}}\backslash\widetilde{Z_\Gamma};\widetilde{\text{Д}_n}\backslash\widetilde{\text{Д}_n}\cap\widetilde{Z_\Gamma})$$

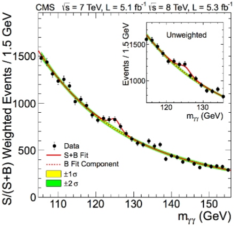
- Z_Γ is the singular locus of the integrand
- $\Delta_n \subset \text{Д}_n := \{x_1 \cdots x_n = 0\}$ is in the normal crossings divisor
- Iterated blowups are needed to separated Z_Γ and Д_n

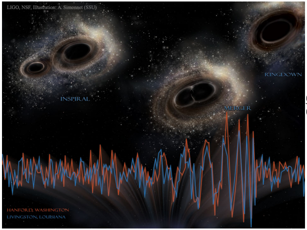


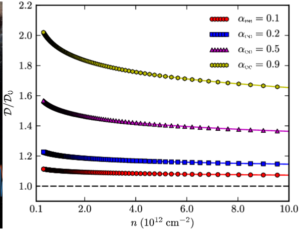
S. Bloch, H. Esnault and D. Kreimer,
On Motives associated to graph polynomials,
Commun. Math. Phys. **267** (2006), 181-225
[arXiv:math/0510011 [math.AG]].

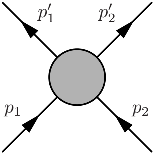
F. Brown,
Feynman amplitudes, coaction principle, and cosmic Galois group,
Commun. Num. Theor. Phys. **11** (2017), 453-556
[arXiv:1512.06409 [math-ph]]

Feynman integrals in physics



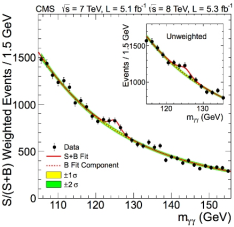


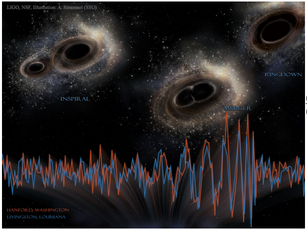


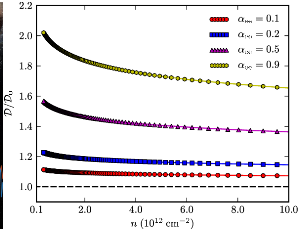


The Feynman integrals enter the evaluation of the scattering amplitudes. They are the fundamental tools for making contact between theoretical physics and experiments

Feynman integrals in physics







Feynman integrals are higher transcendental functions which nature is determined by the singularity structure of the integrand. A classification is not complete but the following large class of functions have been found in gravitational waves computations

- Hyperlogarithm, Elliptic functions, Periods of (singular) K3, (singular) Calabi-Yau Threefold
- The sunset 3-loop $K3$-period appears in the $g-2$ of muon, ...

More is expected as we increase the order of the computations

Feynman Integrals are D-finite functions

Theorem [Kashiwara, Kawai; Petukhov, Smirnov; Bitoun et al.]
Feynman integrals are **holonomic D-finite functions** for generic values of $(D, \nu_1, \ldots, \nu_n)$

$$I_\Gamma(\underline{s}, \underline{m}; \underline{\nu}, D) = \int_{\Delta_n} \frac{\mathcal{U}_\Gamma(\underline{x})^{\nu - \frac{(L+1)D}{2}}}{\mathcal{F}_\Gamma(\underline{x})^{\nu - \frac{LD}{2}}} \prod_{i=1}^{n} x_i^{\nu_i - 1} \Omega_0$$

The vector space

$$V_\Gamma := \sum_{\underline{\nu} \in \mathbb{Z}^n} \mathbb{C}(-D/2, \underline{\nu}) I_\Gamma(\underline{s}, \underline{m}^2; \underline{\nu}, D)$$

has dimension the (signed) Euler characteristic of the complement of the graph hyper-surface [Bitoun, Bogner, Klausen, Panzer]

$$\dim(V_\Gamma) = (-1)^{n+1} \chi\left((\mathbb{C}^*)^n \backslash Z_\Gamma\right),$$

Feynman Integrals differential equations

For a given subset of the physical parameters $\underline{z} := (z_1, \ldots, z_r) \subset \{\underline{s}, \underline{m}^2\}$ we want to derive a Gröbner basis of **minimal order** differential equations

$$\mathcal{L}_\Gamma(\underline{s}, \underline{m}^2, \partial_{\underline{z}}) \int_\sigma \frac{\mathcal{U}_\Gamma(\underline{x})^{\nu - \frac{(L+1)D}{2}}}{\mathcal{F}_\Gamma(\underline{x})^{\nu - \frac{LD}{2}}} \prod_{i=1}^{n} x_i^{\nu_i - 1} \Omega_0 = \mathcal{S}_{\sigma,\Gamma}(\underline{z})$$

We construct differential operators $T_{\underline{z}}$ that annihilate the integrand in cohomology

$$T_{\underline{z}} \Omega_\Gamma = \mathcal{L}_\Gamma(\underline{s}, \underline{m}^2, \underline{\partial_z}) \Omega_\Gamma + d\beta_\Gamma$$

- One could use the GKZ D-module approach but the Feynman integrals are *not generic* toric Euler integrals and the restriction of the D-module is an open difficult question
- We ask that β_Γ is holomorphic on $\mathbb{P}^{n-1} \setminus Z_\Gamma$, i.e. it does not have poles that are not present in Ω_Γ

Pole conditions [Picard (1899)]

Consider the rational function $F(x_1, x_2)$

$$F(x_1, x_2) = \frac{ax_1 + bx_2 + c}{\left(\alpha x_1^2 + \beta x_2^2 + \gamma x_1 x_2 + \delta x_1 + \eta x_2 + \zeta\right)^2} = \sum_{i=1}^{2} \partial_{x_i} \frac{N_i(x_1, x_2)}{D_i(x_1, x_2)}$$

where $a, b, c, \alpha, \beta, \gamma, \delta, \eta, \eta$ are constants

There exists four homogeneous polynomials $N_i(x_1, x_2)$ and $D_i(x_1, x_2)$ with $i = 1, 2$ so that the $F(x_1, x_2)\Omega_0 = d\beta$ is exact

The denominators have poles at $x_2^0 = (a\delta - 2\alpha c)/(2\alpha b - a\gamma)$ which is not a pole of the left-hand-side.

This means one can find a cycle γ passing by x_2^0 such that the integral of $\int_\gamma F(x_1, x_2)$ is finite and non-vanishing.

One could then localise the integral on the residue and work on the reduced cohomology. But this is not easy to do in general as the singularity structure of Feynman integral is quite complicated.

In $D = 2\delta - 2\epsilon$ dimensions with $\delta \in \mathbb{N}$ and $\epsilon \in \mathbb{C}$ we have a twisted differential form

$$\Omega_\Gamma^\epsilon := \left(\frac{\mathcal{U}_\Gamma^{L+1}}{\mathcal{F}_\Gamma^{L}}\right)^\epsilon \frac{\mathcal{U}_\Gamma^{\nu_1+\cdots+\nu_n-\delta(L+1)}}{\mathcal{F}_\Gamma^{\nu_1+\cdots+\nu_n-\delta L}} \prod_{i=1}^n x_i^{\nu_i-1}\Omega_0 \,.$$

This is well defined because $\deg \mathcal{F}_\Gamma = \deg \mathcal{U}_\Gamma + 1 = L + 1$
We consider the partial derivative $\mathrm{a} = a_1 + \cdots + a_r$

$$\left(\frac{\partial}{\partial z_1}\right)^{a_1} \cdots \left(\frac{\partial}{\partial z_r}\right)^{a_r} \Omega_\Gamma^\epsilon = \frac{P^{(a_1,\dots,a_r)}(\underline{x})}{\mathcal{F}_\Gamma^{\mathrm{a}}} \Omega_\Gamma^\epsilon$$

- The locus $\mathcal{F}_\Gamma = 0$ as non-isolated singularities. We need to use syzygies of $\mathrm{Jac}(\mathcal{F}_\Gamma)$
- For $\epsilon \neq 0$ we have twisted differential form

We therefore adapt Griffith's pole reduction for overcome these difficulties.

Extended Griffiths' pole reduction

Reducing $P^{(a_1,\dots,a_r)}(\underline{x})$ in the Jacobian ideal of $\mathcal{F}_\Gamma$

$$P^{(a_1,\dots,a_r)}(\underline{x}) = \vec{C}_{\rm a}(\underline{x}) \cdot \vec{\nabla}\mathcal{F}_\Gamma \,,$$

We introduce the differential twisted form

$$\beta^{(a_1,\dots,a_r)} = \sum_{1\leqslant i<j\leqslant n} \frac{x_i C^j_{\rm a}(\underline{x}) - x_j C^i_{\rm a}(\underline{x})}{\mathcal{F}_\Gamma^{{\rm a}-1}} \, \Omega^\epsilon_\Gamma \, dx_1 \wedge \cdots \wedge \widehat{dx_i} \wedge \cdots \wedge \widehat{dx_j} \wedge \cdots \wedge dx_n$$

Following Griffiths' pole reduction we reduce the pole order of $\mathcal{F}_\Gamma$

$$\frac{P^{(a_1,\dots,a_r)}(\underline{x})}{\mathcal{F}_\Gamma^a}\Omega_\Gamma = \frac{\vec{\nabla}\cdot\vec{C}_{\rm a}(\underline{x}) + \lambda_U \vec{C}_{\rm a}\cdot\vec{\nabla}\log\mathcal{U}_\Gamma}{({\rm a}-1+\lambda_F)\,\mathcal{F}_\Gamma^{{\rm a}-1}}\Omega^\epsilon_\Gamma + \frac{d\beta_\Gamma^{(a_1,\dots,a_r)}}{{\rm a}-1+\lambda_F}$$

where we have defined

$$\lambda_U = n-(L+1)(\delta-\epsilon), \qquad \lambda_F = n - L(\delta-\epsilon).$$

Extended Griffiths' pole reduction

The term $\vec{C}_a \cdot \vec{\nabla} \log \mathcal{U}_\Gamma$ which has a pole in $\mathcal{U}_\Gamma$, which is reduced by asking that

$$\vec{C}_a(\underline{x}) \cdot \vec{\nabla}\mathcal{U}_\Gamma = c_a(\underline{x})\,\mathcal{U}_\Gamma\,,$$

which is equivalent to the computation of syzygies of $\mathrm{Jac}(\mathcal{U}_\Gamma)$ using the homogeneity of $\mathcal{U}_\Gamma$

$$\left(L\vec{C}_a(\underline{x}) - c_a(\underline{x})\vec{x}\right) \cdot \vec{\nabla}\mathcal{U}_\Gamma = 0\,,$$

Solving the linear system

$$\begin{cases} \vec{C}_a(\underline{x}) \cdot \vec{\nabla}\mathcal{F}_\Gamma = P^{(a_1,\dots,a_r)}(\underline{x}) \\ \vec{C}_a(\underline{x}) \cdot \vec{\nabla}\mathcal{U}_\Gamma = c_a(\underline{x})\mathcal{U}_\Gamma \end{cases},$$

we have the pole reduction

$$\frac{P^{(a_1,\dots,a_r)}(\underline{x})}{\mathcal{F}_\Gamma^{a}}\Omega_\Gamma = \frac{\vec{\nabla}\cdot\vec{C}_a(\underline{x}) + \lambda_U\, c_a(\underline{x})}{(a-1+\lambda_F)\mathcal{F}_\Gamma^{a-1}}\Omega_\Gamma^\epsilon + \frac{a-1}{a+n-L(\delta-\epsilon)}d\beta_\Gamma^{(a_1,\dots,a_r)}$$

Extended Griffiths' pole reduction

Phillip A. Griffiths,
On the periods of certain rational integrals: II.
Annals of Mathematics (1969): 496-541.

S. Müller-Stach, S. Weinzierl and R. Zayadeh,
Picard-Fuchs equations for Feynman integrals,
Commun. Math. Phys. **326** (2014), 237-249
arXiv:1212.4389

P. Lairez and P. Vanhove,
Algorithms for minimal Picard-Fuchs operators of Feynman integrals
Lett. Math. Phys. **113** (2023) no.2, 37
arXiv:2209.10962

P. Lairez, E. Pichon-Pharabod and P. Vanhove,
Effective homology and periods of complex projective hypersurfaces
[arXiv:2306.05263]

L. de la Cruz and P. Vanhove,
Algorithm for differential equations for Feynman integrals in general dimensions
[arXiv:2401.09908]

Differential operators I

We turn to the construction of the differential operator

$$\mathcal{L}_\Gamma^\epsilon = \sum_{a=0}^{N(\Gamma,\epsilon)} \sum_{\substack{a=a_1+\cdots+a_r \\ a_i\geqslant 0}} c_{a_1,\dots,a_r}(\vec{m},\vec{s},\epsilon,\kappa) \left(\frac{\partial}{\partial z_1}\right)^{a_1} \cdots \left(\frac{\partial}{\partial z_r}\right)^{a_r}$$

such that

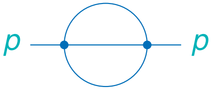
$$\mathcal{L}_\Gamma^\epsilon \Omega_\Gamma^\epsilon = d\beta_\Gamma^\epsilon.$$

We start with the pole reduction with taking a first order derivative, and increase the order until the extended Griffith's pole reduction closes. Holonomicity of Feynman integrals imply that the algorithm finishes because the order $N(\Gamma,\epsilon)$ has for upper bound

$$N(\Gamma,\epsilon) \leqslant \dim(V_\Gamma) = (-1)^{n+1}\chi\left((\mathbb{C}^*)^n \backslash \mathbb{V}(\mathcal{U}_\Gamma) \cup \mathbb{V}(\mathcal{F}_\Gamma)\right)$$

As it turns for many cases we have a strict inequality

The sunset graph in $D = 2 - 2\epsilon$ dimensions



$$I_\ominus^\epsilon(p^2, \underline{m}^2) = \int_{\mathbb{R}_+^3} \left(\frac{\mathcal{U}_\ominus^3}{\mathcal{F}_\ominus^2}\right)^\epsilon \frac{dx_1 dx_2 dx_3}{\mathcal{F}_\ominus(\underline{x})}$$

$$\mathcal{F}_\ominus(\underline{x}) = (x_1x_2 + x_1x_3 + x_2x_3)(m_1^2x_1 + m_2^2x_2 + m_3^2x_3) - p^2x_1x_2x_3$$

- $\mathcal{F}_\ominus(\underline{x}) = 0$ defines an elliptic curve $\mathcal{E}_\ominus$
- For $\epsilon = 0$ the integral is an elliptic dilogarithm in the regulator of a class in the motivic cohomology of the elliptic curve $E_\ominus$

S. Bloch and P. Vanhove,
The elliptic dilogarithm for the sunset graph
J. Number Theor. ***148*** *(2015), 328-364*
[arXiv:1309.5865]

S. Bloch, M. Kerr and P. Vanhove,
Local mirror symmetry and the sunset Feynman integral
Adv. Theor. Math. Phys. ***21*** *(2017), 1373-1453*
arXiv:1601.08181]]

The two-loop sunset graph in general dimensions

$$I_{\ominus}^{\epsilon}(p^2, \underline{m}^2) = \int_{\mathbb{R}_+^3} \left(\frac{\mathcal{U}_{\ominus}^3}{\mathcal{F}_{\ominus}^2}\right)^{\epsilon} \frac{dx_1 dx_2 dx_3}{\mathcal{F}_{\ominus}(\underline{x})}$$

Applying the algorithms we find a fourth order differential equation

$$\mathcal{L}_{\ominus}^{\epsilon} = \mathcal{L}_1^{(1)} \mathcal{L}_1^{(2)} \mathcal{L}_2^{\ominus} + \epsilon \mathcal{L}_4^{(3)} + \epsilon^2 \mathcal{L}_3^{(4)} + \epsilon^3 \mathcal{L}_2^{(5)} + \epsilon^4 \mathcal{L}_1^{(6)} + \epsilon^5 \mathcal{L}_0^{(7)},$$

- The order is the number of irreducible master integrals [Kalmykov, Kniehl] but $4 < \dim(V_{\ominus}) = 7$
- The differential equation is irreducible for generic ϵ which we checked using the algorithms of [Chyzak, Goyer, Mezzarobba]
- The operators $\mathcal{L}_r^{(a)}$ are of order r
- The $\epsilon = 0$ term factorizes $\mathcal{L}_1^{(1)} \mathcal{L}_1^{(2)} \mathcal{L}_2^{\ominus}$

The sunset graph: Gauß-Manin connexion

If one considers a family of elliptic curve E

$$y^2 = 4x^3 - g_2(t)x - g_3(t); \quad j(t) = \frac{g_2(t)^3}{\Delta(t)}; \quad \delta(t) = 3g_3(t)\frac{d}{dt}g_2(t) - 2g_2(t)\frac{d}{dt}g_3(t)$$

the periods satisfy the differential system of equations

$$\frac{d}{dt}\begin{pmatrix}\int_\gamma \frac{dx}{y}\\ \int_\gamma \frac{xdx}{y}\end{pmatrix} = \begin{pmatrix}-\frac{1}{12}\frac{d}{dt}\log\Delta(t) & \frac{3\delta(t)}{2\Delta(t)}\\ -\frac{g_2(t)\delta(t)}{8\Delta(t)} & \frac{1}{12}\frac{d}{dt}\log\Delta(t)\end{pmatrix}\begin{pmatrix}\int_\gamma \frac{dx}{y}\\ \int_\gamma \frac{xdx}{y}\end{pmatrix}$$

The Picard–Fuchs operator acting on the period integral $\int_\gamma dx/y$ is

$$\mathcal{L}_2^E = 144\Delta(t)^2\delta(t)\frac{d^2}{dt^2} + 144\Delta(t)\left(\delta(t)\frac{d\Delta(t)}{dt} - \Delta(t)\frac{d\delta(t)}{dt}\right)\frac{d}{dt}$$
$$+27g_2(t)\delta(t)^3 + 12\frac{d^2\Delta(t)}{dt^2}\delta(t)\,\Delta(t) - \left(\frac{d\Delta(t)}{dt}\right)^2\delta(t) - 12\frac{d\delta(t)}{dt}\Delta(t)\,\frac{d\Delta(t)}{dt}.$$

The two-loop sunset graph in general dimensions

$$I^{\epsilon}_{\ominus}(p^2, \underline{m}^2) = \int_{\mathbb{R}^3_+} \left(\frac{\mathcal{U}^3_{\ominus}}{\mathcal{F}^2_{\ominus}}\right)^{\epsilon} \frac{dx_1 dx_2 dx_3}{\mathcal{F}_{\ominus}(\underline{x})}$$

$$\mathcal{L}^{\epsilon}_{\ominus} = \mathcal{L}^{(1)}_1 \mathcal{L}^{(2)}_1 \mathcal{L}^{\ominus}_2 + \epsilon \mathcal{L}^{(3)}_4 + \epsilon^2 \mathcal{L}^{(4)}_3 + \epsilon^3 \mathcal{L}^{(5)}_2 + \epsilon^4 \mathcal{L}^{(6)}_1 + \epsilon^5 \mathcal{L}^{(7)}_0,$$

- The differential operator $\mathcal{L}^{\ominus}_2$ is the Picard-Fuchs operator for the elliptic curve defined by
- The deformation ϵ affects only the apparent singularities

$$\mathcal{L}^{\epsilon}_{\ominus}\Big|_{\left(\frac{d}{dt}\right)^4} = \Delta(t)\left(-(2\epsilon+5)\,t^2 - 2\left(m_1^2+m_2^2+m_3^2\right)(1+2\epsilon)\,t + (7+6\epsilon)\prod_{i=1}^{4}\mu_i\right)$$

where $\Delta(t) = t^3 \prod_{i=1}^4 (t-\mu_i^2)$ is the discriminant of the elliptic curve $\mathcal{F}_{\ominus} = 0$

Apparent singularities

For a differential equation

$$c_N(z)\frac{d^N f(z)}{dz^N} + \cdots + c_0(z) f(z) = 0,$$

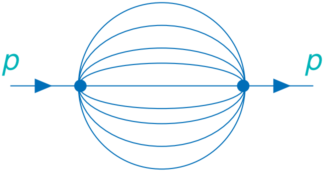
the roots of $c_N(z)$ are the singularities of the differential equation. A root of $c_N(z)$ where the solution $f(z)$ is regular is called an apparent singularity. A root of $c_N(z)$ where the solution has a singularity is a real singularity

For the case of Feynman integrals the non-apparent (real) singularities are the roots of the discriminant of the singular locus of the integrand of Feynman integrals

The parameter ϵ appears only in the apparent singularities of the differential operator $\mathcal{L}_z^\epsilon$. This means that the ϵ deformation does not change the position of the real singularities, but it affects the local behaviour (the monodromy) of the solution near the singularity.

Sunset graph Picard–Fuchs operator

Applying the algorithm to the higher dimensional family of sunset graph



$$\Omega_n^{\ominus}(t,\underline{m}^2) := \frac{\Omega_0}{\mathcal{F}_n^{\ominus}(t,\underline{m}^2;\underline{x})} \in H^{n-1}(\mathbb{P}^{n-1}\backslash X_{\ominus})$$

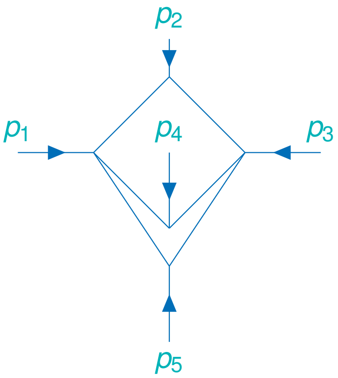
$$\mathcal{F}_n^{\ominus}(t,\underline{m}^2;\underline{x}) := x_1\cdots x_n\left(\left(\sum_{i=1}^{n}\frac{1}{x_i}\right)\left(\sum_{j=1}^{n} m_j^2 x_j\right) - t\right)$$

For generic physical parameters we find a minimal order Picard–Fuchs operator

$$\mathcal{L}_t = \sum_{r=0}^{o_n} q_r(t,\underline{m}^2)\left(\frac{d}{dt}\right)^r \qquad o_n = 2^n - \binom{n+1}{\lfloor\frac{n+1}{2}\rfloor};\, n \geqslant 2.$$

supporting that we have relative periods of a Calabi–Yau of dimension $n-2$.

Tardigrade



The rational differential form in $\mathbb{P}^5$

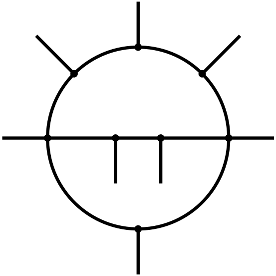
$$\Omega(t) = \frac{\Omega_0^{(6)}}{\left(\mathcal{U}_6(\underline{x})\mathcal{L}_6(\underline{m}^2, \underline{x}) - t\mathcal{V}_6(\underline{s}, \underline{x})\right)^2}$$

$$\mathcal{U}_6(\underline{x}) = (x_1 + x_2)(x_3 + x_4) + (x_1 + x_2)(x_5 + x_6) + (x_3 + x_4)(x_5 + x_6)$$

$\mathcal{V}_6(\underline{s}, \underline{x}) = \sum_{1\leqslant i,j,k\leqslant 6} C_{ijk} y_i y_j y_k$ with linear changes $(x_{2i-1}, x_{2i}) \to (y_{2i-1}, y_{2i})$ and $i = 1, 2, 3$ C_{ijk} symmetric traceless i.e. $C_{iij} = 0$

- The algorithm gives an irreducible Picard–Fuchs operator of order 11 with an head polynomial of degree up to 215
- We have confirmed that this is K3 an elliptically fiber K3 with singular fibres $14I_1 \oplus 2I_4 \oplus I_2$ of Picard number 11

Motives for two-loop graphs



For two-loop Feynman graphs of (a, b, c) vertices we consider the differential form

$$\Omega_{(a,b,c);D} = \frac{\mathcal{U}_{(a,b,c)}^{a+b+c-\frac{3D}{2}}}{\mathcal{F}_{(a,b,c);D}^{a+b+c-D}} \Omega_0$$

where $\deg \mathcal{U}_{(a,b,c)} = 2$ and $\deg \mathcal{F}_{(a,b,c);D} = 3$

We determine the mixed Hodge structure

$$\mathcal{H}^{a+b+c-1} := H^{a+b+c-1}(\mathbb{P}^{\widetilde{a+b+c}-1} \backslash \widetilde{Z}_{(a,b,c)}; \widetilde{\Pi}_n \backslash \widetilde{\Pi}_n \cap \widetilde{Z}_{(a,b,c)})$$

The method is based on quadric fibrations. The cohomology of $Z_{(a,b,c)}$ is obtained by iterated extensions with Tate twists of cohomology of hyperelliptic curves and Tate Hodge structure

We have presented some examples $(a, b, c) = (1, 1, 1)$ sunset, and $(a, b, c) = (2, 2, 2)$ tardigrade

Definitions

Definition

1. Let $\mathbf{MHS}_{\mathbb{Q}}$ denote the abelian category of $\mathbb{Q}$-mixed Hodge structures.
2. The largest extension-closed subcategory of $\mathbf{MHS}_{\mathbb{Q}}$ containing the Tate twists of $\mathrm{H}^1(C;\mathbb{Q})$ for every hyperelliptic curve C is called $\mathbf{MHS}_{\mathbb{Q}}^{\mathrm{hyp}}$.
3. The largest extension-closed subcategory of $\mathbf{MHS}_{\mathbb{Q}}$ containing the Tate twists of $\mathrm{H}^1(E;\mathbb{Q})$ for every elliptic curve E is called $\mathbf{MHS}_{\mathbb{Q}}^{\mathrm{ell}}$.

We have the following results for the Hodge structure of two-loop planar graphs

C. F. Doran, A. Harder, E. Pichon-Pharabod and P. Vanhove,
Motivic geometry of two-loop Feynman integrals
[arXiv:2302.14840]

Graph $(a, 1, c)$

Theorem

1. *If* $3D/2 \leqslant a + c$ *then* $\mathcal{H}^{a+c} \in \mathbf{MHS}_{\mathbb{Q}}^{\text{hyp}}$.
2. *Suppose* $a \leqslant 2$ *or* $c \leqslant 2$. *Then* $\mathcal{H}^{a+c-1}(Z_{(a,1,c)}; \mathbb{Q}) \in \mathbf{MHS}_{\mathbb{Q}}^{\text{ell}}$.

Corollary

If $3D/2 \leqslant a + c$ *and either* $a \leqslant 2$ *or* $c \leqslant 2$ *then* $\mathcal{H}^{a+c} \in \mathbf{MHS}_{\mathbb{Q}}^{\text{ell}}$.

This means that the mixed Hodge structure $\mathrm{H}^{a+c}(\mathbb{P}_\Gamma - \widetilde{Z}_{(a,1,c)}; B - (B \cap \widetilde{Z}_{(a,1,c)}))$ is constructed by taking iterated extensions of $\mathrm{H}^1(E; \mathbb{Q})(-a)^{r_1}$ and $\mathbb{Q}(-b)^{r_2}$ for different values of a, b, r_1, and r_2, and with various possibly different elliptic curves.
Therefore the Feynman integrals in this cases are built from algebraic and elliptic functions.

Graph $(3,1,3)$ double-box

$$\Omega_{(3,1,3);D}(t) = \frac{\mathcal{U}_{(3,1,3)}(\underline{x})\Omega_0}{(\mathcal{U}_{(3,1,3)}(\underline{x})\mathcal{L}_{(3,1,3)}(\underline{m}^2,\underline{x}) - t\mathcal{V}_{(3,1,3);D}(\underline{s},\underline{x}))^3}$$

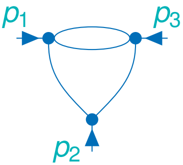
Theorem

For arbitrary kinematic parameters, and arbitrary space-time dimension D, $W_4\mathrm{H}^5(Z_{(3,1,3)};\mathbb{Q})$ is mixed Tate.

1. *If $D \geqslant 5$ then $\mathrm{Gr}_5^W\mathrm{H}^5(Z_{(3,1,3);D};\mathbb{Q}) \cong \mathrm{H}^1(C;\mathbb{Q})(-2)$ for a curve C which has genus 2 for generic kinematic parameters.*
2. *If $D = 4$ then $\mathrm{Gr}_5^W\mathrm{H}^5(Z_{(3,1,3)};\mathbb{Q}) \cong \mathrm{H}^1(E;\mathbb{Q})(-2)$ for a curve E which is elliptic for generic kinematic parameters.*
3. *If $D \leqslant 4$ then $\mathrm{H}^5(Z_{(3,1,3)};\mathbb{Q})$ is mixed Tate.*

In $D = 4$ the PF operator obtained by the extended Griffith–Dwork construction is *identical* to the one associated with the canonical differential form on the elliptic curve defined from the graph polynomial

Graph $(1,1,2)$ ice-cream



With a regular β the Picard–Fuchs operator is of order 2 and degree 9

$$\mathcal{L}_t^2 = q_0(t) + q_1(t)\frac{d}{dt} + q_2(t)\left(\frac{d}{dt}\right)^2,$$

$\mathcal{L}_t$ is a Liouvillian differential equations with only rational solutions

Proposition

- $\mathrm{H}^2(Z_{(2,1,1)}(t);\mathbb{Q})$ *is generically pure Tate.*
- *Let* $Z = V(Disc(Z_{(2,1,1)})(x,z,t)) \subseteq \mathbb{P}^1 \times \mathbb{A}^1$ *and we may define* $\mathbb{V} := \pi_* \underline{\mathbb{Q}}_Z$. *The local system* $\mathbb{V}$ *is isomorphic to the direct sum* $\mathbb{U}_1 \oplus \mathbb{U}_2 \oplus \underline{\mathbb{Q}}_B^2$ *where* $\mathbb{U}_1$ *is a rank 1 local system and* $\mathbb{U}_2$ *is a rank 2 local system*
- *For generic kinematic and mass parameters,* $Sol(\mathcal{L}_{(2,1,1)})$ *is isomorphic to* $\mathbb{U}_2 \cong \mathbb{U}_2^\vee$.

Graph $(1,1,2)$ ice-cream : Picard–Fuchs operator

Lemma

Let $\mathbb{L}_1, \mathbb{L}_2$ *be local systems of rank 1, and suppose that* $\mathbf{s}_1$ *and* $\mathbf{s}_2$ *are sections of* $\mathbb{L}_1 \otimes \mathcal{O}_M$ *and* $\mathbb{L}_2 \otimes \mathcal{O}_M$ *respectively. If*

$$\mathcal{L}_{\mathbf{s}_1} = \frac{d}{ds} - f_1(s), \qquad \mathcal{L}_{\mathbf{s}_2} = \frac{d}{ds} - f_2(s)$$

are the differential equations associated to $\mathbf{s}_1$ *and* $\mathbf{s}_2$ *respectively, then the differential equation associated to the section* $\mathbf{s}_1 \oplus \mathbf{s}_2$ *of* $(\mathbb{L}_1 \oplus \mathbb{L}_2) \otimes \mathcal{O}_M$ *is*

$$\mathcal{L}_{\mathbf{s}_1 \oplus \mathbf{s}_2} = (f_1(s) - f_2(s)) \frac{d^2}{ds^2} + (f_2(s)^2 - f_1(s)^2 - f_1'(s) + f_2'(s)) \frac{d}{ds} + f_2(s) f_1'(s) - f_1(s) f_2'(s) + f_1(s)^2 f_2(s) - f_1(s) f_2(s)^2$$

Graph $(1,1,2)$ ice-cream : Picard–Fuchs operator

Proposition

After the base change

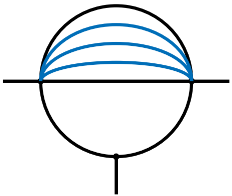
$$t = \frac{(m_1 - m_2)^2 s^2 + (m_1 + m_2)^2}{p_2^2 (s^2 + 1)},$$

the differential operator $\mathcal{L}_{(2,1,1);2}$ *is of the form given previously where* $f_i(s)$ *with* $i = 1, 2$ *are obtained from the application of the change of variables* $T = \rho_i(s)$ *with* $i = 1, 2$ *to the differential operator*

$$\frac{d}{dT} - \frac{(m_3^2 + m_4^2 - T)}{((m_3 - m_4)^2 - T)((m_3 + m_4)^2 - T)} \Longrightarrow \frac{d}{ds} - f_i(s)$$

with $T = \rho_i(s)$ *and* $i = 1, 2$ *are roots of discriminant obtained from blowing up the linear subspace* $x_1 = z = 0$

multiscoop ice-cream



For the multiscoop ice cream cone families there is a conic fibration on $X_{(2,[1]^k);D}$, and the discriminant locus this conic fibration is a union of two Calabi–Yau $(k-2)$-folds associated to the $(k-1)$-loop sunset graph

Therefore $\mathrm{Gr}^W_k \mathrm{H}^k(Z_{(2,[1]^k)};\mathbb{Q})$ arises from

$$\mathrm{H}^{k-2}(Z^{(1)}_{([1]^k)};\mathbb{Q}) \oplus \mathrm{H}^{k-2}(X^{(2)}_{([1]^k)};\mathbb{Q})$$

where $Z^{(1)}_{([1]^k)}$ and $Z^{(2)}_{([1]^k)}$ are distinct $(k-1)$-loop sunset Calabi–Yau $(k-2)$-folds. This is supported by the computations of the PF operator for the 2-scoop ice-cream which is of rank 4. In this case $Z^{(1)}_{(1,1,1)}$ and $Z^{(2)}_{(1,1,1)}$ are elliptic curves, so the rank of $\mathcal{L}_{(2,1,1,1)}$ agrees with the rank of $\mathrm{H}^1(Z^{(1)}_{(1,1,1)};\mathbb{Q}) \oplus \mathrm{H}^1(Z^{(2)}_{(1,1,1)};\mathbb{Q})$.

Conclusion

We have presented that an extension of the Griffiths-Dwork algorithm for computing differential operators for Feynman integrals

- The algorithm work for non-smooth case (which is the generic case for Feynman integral)
- The algorithm works for twisted cohomology so we get the D-module of differential operators for the relative periods

Todo list: with Spencer and Matt we had evaluated the two- and three-loop sunset integrals for $\epsilon = 0$ as a regulator in $K_2(E_\ominus)$ or $K_3(K3)$ respectively. What is the equivalent statement for twisted cohomology.