Atypical Hodge Loci

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Outline

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*Talk based on the paper [BKU] and related works given in the references in that work, and on extensive discussions with Mark Green and Colleen Robles. The last section has benefitted through discussions with Matt Kerr.

I. Introduction

Q: *What can one say about Hodge loci?*

- B is a smooth, connected quasi-projective variety;
- $\mathbb{V} \to B$ is the local system underlying a variation of polarized Hodge structure of weight n;
- geometric case; smooth projective family $\mathcal{X} \xrightarrow{\pi} B$ and

$$\mathbb{V}_b = H^n(X_b, \mathbb{Q})_{\mathrm{prim}};$$

- $\mathrm{HL}(B) =$ set of $b \in B$ with more Hodge classes in $\mathbb{V}_b^{\otimes} := \overset{k}{\oplus}(\overset{k}{\otimes}\,\mathbb{V}_b)$ than at a very general point of b;
- Cattani-Deligne-Kaplan: $\mathrm{HL}(B)$ is a countable union of proper algebraic subvarieties;
- in geometric case assuming the Hodge conjecture there are extra classes of algebraic cycles in $\underbrace{X \times \cdots \times X}_{k}$'s.

Q: *What can we say about* $\mathrm{HL}(B)$*?*

- very informally stated the main result in [BKU] is

(I.1) *For* $n \geqq 3$ *and aside from exceptional degenerate cases, every irreducible component of* $\mathrm{HL}(B)_{\mathrm{pos}}$ *has strictly larger than the expected codimension;*

- know of no conceptual reason why in the non-classical case there should be more than the expected amount of algebraic cycles;
- proof uses integrability conditions for the differential constraint imposed by transversality in the non-classical case;
- sufficient condition for result is

$$\mathfrak{g}^{-k,k} \neq 0, \qquad \text{some } \ k \geqq 3;$$

- notation and criterion for this given below.
- implied by coupling length $\geqq 3$

II. Two examples

- $X = X_b$, $T = T_b B$ and $T \to H^1(T_X) = T\,\mathrm{Def}(X)$;
- $V^{p,q} = H^q(\Omega_X^p)$ and $T \to \oplus\,\mathrm{Hom}(V^{p,q}, V^{p-1,q+1})$ is Kodaira-Spencer mapping giving first variation of Hodge decomposition of a class in $H^n(X)$;
- for X a surface, $\theta \in T$, $\lambda \in \mathrm{Hg}^1(X)$ and $\theta \cdot \lambda \in H^{0,2}(X)$ gives the first order deviation from λ remaining a Hodge class in the direction θ;
- $\mathrm{NL}_\lambda \subset B$ is the Noether-Lefschetz locus where λ remains a Hodge class; assume reduced and define $T_\lambda \subset T = \ker\{\theta \to \theta \cdot \lambda\}$;
- for $X \subset \mathbb{P}^3$ of degree $d \geqq 4$ in the estimate

$$d - 3 \leqq \operatorname{codim}_B \mathrm{NL}_\lambda \leqq \binom{d-1}{3} = h^{2,0}(X)$$

both bounds are achieved (Green; lower bound $\iff$ X contains a line);

- now let $\dim X = 4$, $\lambda \in \mathrm{Hg}^2(X)_{\mathrm{prim}}$; in first approximation

$$\operatorname{codim} \mathrm{NL}_\lambda \leqq h^{1,3}(X) + h^{0,4}(X);$$

- but $\theta \cdot \lambda \in H^{1,3}(X)$ so this estimate must be refined to

$$(*) \qquad \operatorname{codim}_B \mathrm{NL}_\lambda \leqq h^{1,3}(X).$$

Definition: The right-hand side of $(*)$ is the *expected codimension* of NL_λ in B.

- *Integrability:* With $T_\lambda \subset T$ as above set

$$\sigma_\lambda = \mathrm{Image}\{T_\lambda \otimes H^{4,0}(X) \to H^{3,1}(X)\}.$$

Observation:

$$\operatorname{codim}_B \mathrm{NL}_\lambda \leqq h^{1,3}(X) - \dim \sigma(\lambda).$$

Proof: For $\theta \in T_\lambda$, $\theta' \in T$, $\omega \in H^{4,0}(X)$

$$\begin{aligned}\langle \theta\omega, \theta'\lambda\rangle &= \langle \omega, \theta\theta'\lambda\rangle \\ &= \langle \omega, \theta'\theta\lambda\rangle \quad \text{(integrability)} \\ &= 0.\end{aligned}$$

Thus the number of conditions on $\theta' \in T$ to be in T_λ is $\leqq h^{1,3} - \dim \sigma(\lambda)$. □

- **Note:** For the first example of $X \subset \mathbb{P}^3$ the expected codimension drops for geometric reasons: if $L \subset X$ is a line with Hodge class λ and if $\omega \in H^0(\Omega_X^2) \cong H^0(\mathcal{O}_X(d-4))$, then if $L \subset (\omega)$,

$$\langle \theta\lambda, \omega\rangle = \langle \lambda, \theta \cdot \omega\rangle = \int_L \theta \rfloor \omega = 0$$

for all $\theta \in H^1(T_X)$; thus such ω's do not contribute to the equations defining NL_λ. In the second example the drop by σ_λ in the expected codimension is for Hodge theoretic reasons.

III. Statement of main result

- *Polarized Hodge structure* $(V, Q, F^\bullet)$ *of weight* n
  - non-degenerate $Q : V \otimes V \to \mathbb{Q}$, $Q(u, v) = (-1)^n Q(v, u)$;
  - $F^n \subset F^{n-1} \subset \cdots \subset F^0 = V_{\mathbb{C}}$, $F^p \oplus \overline{F}^{n-p+1} \xrightarrow{\sim} V_{\mathbb{C}}$ for $0 \leqq p \leqq n$;
  - $V^{p,q} = F^p \cap \overline{F}^q$, $V_{\mathbb{C}} = \oplus V^{p,q}$ with $\overline{V^{p,q}} = V^{q,p}$;
  - Hodge-Riemann bilinear relations;
  - $n = 2m$, $\mathrm{Hg}^m(V) = V^{m,m} \cap V$.
  - Lie algebra $\mathfrak{g} \subset \mathrm{End}(V, Q)$ and $\mathfrak{g}_{\mathbb{C}} = \oplus \mathfrak{g}^{-k,k}$ where

$$\mathfrak{g}^{-k,k} := \mathfrak{g}^{-k} = \{A \in \mathfrak{g}_{\mathbb{C}} : A(V^{p,q}) \subset V^{p-k,q+k}\}$$

$$\ell(\mathfrak{g}) = \min\{k : \mathfrak{g}^{-k} \neq (0)\};$$

  - $\ell(\mathfrak{g}) \geqq 3 \implies n \geqq 3$.

- *Mumford-Tate group*
  - $V^{\otimes} := \overset{k}{\oplus}(\otimes^k V)$;
  - Hodge tensors $\mathrm{Hg}^{\bullet}(V) = \overset{k}{\oplus}\,\mathrm{Hg}^{kn/2}(V)$;
  - $\mathrm{MT}(V) \subset \mathrm{Aut}(V, Q)$ is $\mathrm{Fix}(\mathrm{Hg}^{\bullet}(V))$;
  - is a reductive $\mathbb{Q}$-algebraic group H;
  - finite cover of H is $\mathbb{C}^{*k} \times H_0$ where H_0 is semi-simple; for simplicity of exposition we will assume H is semi-simple; essential ideas appear in this case;

Example: Assume λ, Q generate the algebra of Hodge tensors, $\mathrm{MT}(V) = H_\lambda = \mathsf{Fix}\ \lambda \subset \mathrm{Aut}(V, Q)$.

- *Variation of Hodge structure* $(\mathbb{V}, \mathcal{F}^\bullet; B)$
  - B and $\mathbb{V} \to B$ as above;
  - $\mathcal{F}^\bullet$ is a filtration of $\mathcal{V} := \mathbb{V}_{\mathbb{C} \otimes \mathbb{C}} \mathcal{O}_B$ inducing a polarized Hodge structure on each $\mathbb{V}_b$ (understood there is $Q : \mathbb{V} \otimes \mathbb{V} \to \mathbb{Q}$);
  - $\nabla \mathcal{F}^p \subset \mathcal{F}^{p-1} \otimes \Omega^1_B$ (transversality);
  - for $b_0 \in B$ and $V = V_{b_0}$ we have the monodromy group $\Gamma \subset \mathrm{Aut}(V, Q)$;
  - the $\mathbb{Q}$-Zariski closure $\overline{\Gamma^{\mathbb{Q}}}$ = semi-simple $\mathbb{Q}$-algebraic group that is a factor of the MT-group of $(V, \mathbb{Q}, F^\bullet_b)$ at a very general point of B.

- *Period mappings*
  - G = semi-simple $\mathbb{Q}$-algebraic group and D = period domain of polarized Hodge structures of a given type and with generic Mumford-Tate group G;
  - $D = G(\mathbb{R})/G_0$, G_0 compact;
  - period mapping $\Phi : B \to \Gamma\backslash D$;
  - for simplicity of exposition we will assume that $\overline{\Gamma}^{\mathbb{Q}} := G$ =Mumford-Tate group of the polarized Hodge structure at a very general point of B;

  This is justified because the image $\Phi(B)$ of the period mapping is contained in a translate of a $\overline{\Gamma^{\mathbb{Q}}}(\mathbb{R})$-orbit.

  - $T_{F_0}D \cong \mathfrak{g}_{\mathbb{C}}/F^0\mathfrak{g}_{\mathbb{C}} \cong \overset{k>0}{\oplus}\ \mathfrak{g}^{-k,k}$ where $\mathfrak{g}_{\mathbb{C}} \subset \mathrm{End}(V_{\mathbb{C}})$ and $\theta \in \mathfrak{g}^{-k,k}$ satisfies $\theta(F^p) \subset F^{p-k}$;
  - $\Phi_* : T_bB \to \mathfrak{g}^{-1,1}$ (transversality);
  - image is an abelian subalgebra $\mathfrak{a} \subset \mathfrak{g}^{-1,1}$ (integrability);
  - notation: $\Phi(B) = P \subset \Gamma\backslash D$, $\widetilde{P} \subset D$ inverse image of P.

- *Hodge loci*
  - if λ, Q generate $\mathrm{Hg}^\bullet(V_b^{\otimes})$, then

$$\mathrm{NL}_\lambda^0 = \Phi^{-1}(\Phi(B) \cap G_\lambda(\mathbb{R}) \cdot o)^0 \qquad (^0 = \text{identity component});$$

  - thus Noether-Lefschetz loci are (translates of) orbits of particular Mumford-Tate subgroups of G;
  - this is the crucial conceptual point; **Hodge loci are intersections with translates of Mumford-Tate sub-domains.**

Definition: If $H \subset G$ is a Mumford-Tate group and

$$D_H = H(\mathbb{R}) \cdot o, \qquad \Gamma_H = \Gamma \cap H$$

$\Phi^{-1}(\Phi(B) \cap (\Gamma_H \backslash D_H))^0$ is a **special subvariety** of B.

- thus special subvarieties of B are those $b \in B$ where the algebra of Hodge tensors is strictly larger than at a general point of B.

- set $P_H = P \cap (\Gamma_H \backslash D_H)$, then the standard codimension of an intersection inequality is

$$(\natural) \quad \mathrm{codim}_{\Gamma\backslash D} P_H \leqq \mathrm{codim}_{\Gamma\backslash D} P + \mathrm{codim}_{\Gamma\backslash D}(\Gamma_H \backslash D_H);$$

Definition: Special subvariety is **atypical** if we have strict inequality in $(\natural)$.

- Thus atypical means we have more Hodge tensors than predicted by the usual expected dimension count formula.

Example: Notations as in the second example above

$$\mathrm{codim}_{\Gamma\backslash D}(\Phi(B) \cap (\Gamma_\lambda \backslash D_\lambda)) = \mathrm{codim}_{\Gamma\backslash D} \Phi(B) + \mathrm{codim}_{\Gamma\backslash D}(\Gamma_\lambda \backslash D_\lambda) - \dim \sigma(\lambda).$$

Theorem ([BKU]): *If $\ell(\mathfrak{g}) \geqq 3$, then* **every** *special subvariety of B is atypical.*

Example: For smooth $X \subset \mathbb{P}^{n+1}$, $n \geqq 3$ and $\deg(X) \geqq 6$ every special subvariety is atypical.

Reason for result: M is a manifold and I', I'' distributions in TM given by $\{\omega'_i\}$, $\{\omega''_\alpha\}$; N', N'' variable integral manifolds of $I', I''|_{N'}$'s; want to estimate codimension of the $N' \cap N''$'s; integrability conditions given by $d\omega'_i$ may impose linear relations on the $\omega''_\alpha|_{N'}$; leads to more than expected number of $N' \cap N''$'s.

IV. Proof of the main result

- Assume equality holds in (♮) and will arrive at a contraction to the assumption $\mathfrak{g}^{-k} \neq 0$;
- pass to tangent spaces and use

$$\dim T_0 D = \mathfrak{g}^-,$$
$$\dim T_0 D_H = \mathfrak{h}^-,$$
$$T_0 \widetilde{P} \subset \mathfrak{g}^{-1},$$
$$T_0(\widetilde{P} \cap D_H) = T_0 \widetilde{P} \cap \mathfrak{h}^{-1}$$

$$\sum_{k \geqq 2} \dim \mathfrak{h}^{-k} + \mathrm{codim}_{\mathfrak{h}^{-1}}(T_0 \widetilde{P} \cap \mathfrak{h}^{-1}) = \textstyle\sum_{k \geqq 2} \dim \mathfrak{g}^{-k} + \mathrm{codim}_{\mathfrak{g}^{-1}}\, T_0 \widetilde{P}$$
$$\dim \mathfrak{h}^{-k} \leqq \dim \mathfrak{g}^{-k}$$
$$\mathrm{codim}_{\mathfrak{h}^{-1}}(T_0 \widetilde{P} \cap \mathfrak{h}^{-1}) \leqq \mathrm{codim}_{\mathfrak{g}^{-1}}\, T_0 \widetilde{P}$$
$$\Longrightarrow \begin{cases} \mathfrak{h}^{-k} = \mathfrak{g}^{-k}, & k \geqq 2 \\ \mathrm{codim}_{\mathfrak{h}^{-1}}\, T_0 \widetilde{P} \cap \mathfrak{h}^{-1} = \mathrm{codim}_{\mathfrak{g}^{-1}}\, T_0 \widetilde{P}. \end{cases}$$

We want to conclude

$$(\natural\natural) \qquad \begin{array}{ccc} \mathfrak{h}^{-1} & = & \mathfrak{g}^{-1} \\ & \Downarrow & \\ T_0 D_H & = & T_0 D \implies D_H = D. \end{array}$$

- *Basic idea*
  - $\mathfrak{h}, \mathfrak{g}$ are reductive Lie algebras in which $\mathfrak{h}^-, \mathfrak{g}^-$ are parabolic sub-algebras;
  - there is maximal torus $\mathfrak{t}_\mathbb{R} \subset \mathfrak{g}_\mathbb{R}$ relative to which $\mathfrak{h}^-, \mathfrak{g}^-$ are direct sums of negative root spaces;
  - the root space and Hodge decomposition of $\mathfrak{h}^-, \mathfrak{g}^-$ align in the sense that each of $\mathfrak{h}^{-k}, \mathfrak{g}^{-k}$ are direct sums of negative root spaces;
  - this is the first key point where the Lie theory and Hodge theory interact; the complex structure on $T_0 D \cong \mathfrak{g}_\mathbb{R}/\mathfrak{g}^0_\mathbb{R}$ is given by an $E \in \mathfrak{t}_\mathbb{R}$ whose centralizer is $\mathfrak{g}^0_\mathbb{R}$ and whose eigenspace decomposition on $\mathfrak{g}_\mathbb{R}/\mathfrak{g}^0_\mathbb{R}$ is a direct sum of non-trivial conjugate root subspaces (cf. [R] for details);

- let β_i, $i \in I$, be the simple positive roots whith corresponding root space $\mathfrak{g}_{\beta_i} \subset \mathfrak{g}^+$; denote by $J \subset I$ the subset, possibly empty, where $\mathfrak{g}_{\beta_i} \subset \mathfrak{h}^+$; since every positive root is a sum of the β_i, it will suffice to show that

$$(\flat) \qquad J = I;$$

in this case we will have $(\natural\natural)$;
- note that $\mathfrak{g}^-$ is generated by $\mathfrak{g}^{-1} \iff$ all $\mathfrak{g}_{-\beta_i} \subset \mathfrak{g}^{-1}$.

- *Second key point*
  - this is where integrability comes in; $\widetilde{P} \subset D$ is an integral manifold of the $G(\mathbb{R})$-invariant distribution of $T(D)$ given by $\mathfrak{g}^{-1} \subset \mathfrak{g}^- = T_0 D$; the real Lie algebra generated by the brackets in $\mathfrak{g}^{\pm 1}$ corresponds to a reductive subgroup $G'_{\mathbb{R}} \subset G(\mathbb{R})$ and $\widetilde{P} \subset G'_{\mathbb{R}} \cdot o$; thus we may assume

$$\mathfrak{g}^{-1} \text{ bracket generates } \mathfrak{g}^-$$

(cf. [R] for details);

- suitably interpreted the previous considerations apply also to $\widetilde{P} \cap D_H$; the upshot is that in effect we may assume this bracket generating property also for $\mathfrak{h}^{-1}$ and $\mathfrak{h}^{-}$;
- we now assume ($\flat$) also does not hold, and from this note that

$$(\flat\flat) \qquad [\mathfrak{g}_{\beta_i} \cdot \mathfrak{g}_{\beta_j}] = 0, \quad j \in J \text{ and } i \in I \backslash J.$$

Indeed, if this bracket is non-zero, then since $\mathfrak{h}^{-2} = \mathfrak{g}^{-2}$ it belongs to $\mathfrak{h}^2$ and the non-zero root space

$$\mathfrak{g}_{\beta_i} = [[\mathfrak{g}_{\beta_i}, \mathfrak{g}_{\beta_j}], \mathfrak{g}_{-\beta_j}] \in \mathfrak{h}^1$$

which is a contradiction;

- for the final step, if $\mathfrak{g}^3 \neq 0$, we have $\beta_1, \beta_2, \beta_3$ such that $\beta_1 + \beta_2 + \beta_3$ is a root. Then

$$[\mathfrak{g}_{\beta_1+\beta_2+\beta_3}, \mathfrak{g}_{-\beta_1-\beta_2}] = \mathfrak{g}_{-\beta_3} \in \mathfrak{h}^1;$$

  thus

$$J \neq \emptyset.$$

  If $J \neq I$, then (♭♭) gives a contradiction to the fact that the highest root is $\sum_{i \in I} n_i \beta_i$ with all $n_i > 0$.
- If $n = 2$, the argument works all the way up to the last step where, as in the first example, we do have $J = \emptyset$ (and $\mathfrak{g}^{-2} \neq 0 \iff h^{2,0} > 0$ for $G = \mathrm{SO}(2a, b)$).

- *Examples:* $X \subset \mathbb{P}^{n+1}$ smooth degree d hypersurface

$$F(x) = 0$$

where $F(x)$ homogeneous of degree d. For

$$\begin{cases} S^\bullet = \mathbb{C}[x_0, \cdots, x_{n+1}], \\ J^\bullet = \text{Jacobian ideal } \{F_{x_0}, \cdots, F_{x_{n+1}}\}, \\ R^\bullet = S^\bullet / J_F^\bullet \end{cases}$$

there is an isomorphism

$$H^{p,n-p}(X)_{\mathrm{prim}} \cong R^{(n-p)d+n-2};$$

tangent space family of X's is

$$T \cong R^d$$

and

$$(\sharp\sharp) \qquad T \otimes H^{p,n-p}(X)_{\mathrm{prim}} \to H^{p-1,n-p+1}(X)_{\mathrm{prim}}$$

given by multiplication of polynomials

$$R^d \otimes R^{(n-p)d+n-2} \to R^{(n-p+1)d+n-2}.$$

X non-singular gives (Macauly's theorem) that

> *the mappings* $(\sharp\sharp)$ *are non-zero whenever both sides are non-zero.*

G is Mumford-Tate group for the period mapping of X's, then

$$R^d \to \mathfrak{g}^{-1,1} \subset F^{-1}\,\mathrm{End}(V, Q).$$

Image is an *abelian* sub-algebra $\mathfrak{a} \subset \mathfrak{g}^{-1.1} \subset \mathfrak{g}_{\mathbb{C}}$, induces

$$\mathrm{Sym}^k \mathfrak{a} \to \mathfrak{g}^{-k,k}$$

giving

$$\mathrm{Sym}^k \mathfrak{a} \to \mathfrak{g}^{-k,k} \subset \oplus \mathrm{Hom}\left(H^{p,n-p}(X)_{\mathrm{prim}}, H^{p-k,n-p+k}(X)_{\mathrm{prim}}\right)$$

which is a subspace of

$$\mathrm{Sym}^k R^d \otimes R^{(n-p)d+n-2} \to R^{(n-p+k)d+n-2}$$

given by multiplication of polynomials. Conclude that map is non-zero whenever both sides are non-zero, which gives $\mathfrak{g}^{-3,3} \neq 0$ for $n \geqq 3, d \geqq 6$. □

- In the second example at the beginning

$$\sigma(\lambda) = 0 \iff \mathfrak{g}^{-3} = (0).$$

- in general *coupling length* defined by

$$\zeta(\mathfrak{a}) = \max\{m : \mathrm{Sym}^m \mathfrak{a} \to \mathrm{Hom}(\mathbb{V}_b^{n,0}, \mathbb{V}_b^{n-m,m}) \neq 0\}$$

at a general point of B. Then

$$\zeta(\mathfrak{a}) \geqq \ell(\mathfrak{g}).$$

There are many examples where $\zeta(\mathfrak{a}) \geqq 3$; e.g., hypersurfaces as above, CY's of dimension $\geqq 3$ whose Yukaya coupling is $\neq 0$.

V. Motivic Hodge structures

- Recent posting arXiv.org/abs/2308.16164 by Tobias Kreutz gives an interesting application of the method in [BKU].
- Idea is nice; statement of the result is not complete because it does not use integrability of transversality; following is an amended version.
- Polarized Hodge structure (PHS) $(V, F^\bullet, Q) := H$ *comes from geometry* if
  - (first approximation) $H = H^n(X)$ for a smooth projective variety X,
  - actual definition is motivic; basically H is made up of sub-quotients of the above.
- These objects have Mumford-Tate groups G and corresponding Mumford-Tate domains D with compact dual $\check{D} = G(\mathbb{C})/P$; this is a homogenous algebraic variety defined over $\overline{\mathbb{Q}}$.

- If D is non-classical then most points of D do not come from geometry; intuitive reason is that because of the differential constraint the image of a period mapping does *not* contain an open set; the set of points of coming from geometry is the complement of a countable union of proper analytic subvarieties.
- Nobody has exhibited an explicit H not coming from geometry; assuming the generalized Hodge conjecture (GHC) and the version due to André of Grothendieck's generalized period conjecture (GPC), Kreutz gives a necessary condition that H come from geometry.
- With terms to be explained the result is

$$\boxed{\operatorname{tr}\deg(H) < L(\mathfrak{g}) \implies H \text{ does not come from geometry.}}$$

- H is defined over a field k if equivalently
  - $F^p \subset V \otimes_{\mathbb{Q}} k$,
  - $F^\bullet \in \check{D}(k)$.

- Then the definition

$$\operatorname{tr}\deg(H) := \min \operatorname{tr}\deg(k)$$

makes sense.

- As above $H \in \check{D} = G(\mathbb{C})/P$ where $G =$ Mumford-Tate group of H, and we define

$$L(\mathfrak{g}) := \min\left\{\operatorname{codim}_{\mathfrak{g}/\mathfrak{g}^0} \mathfrak{a} : \mathfrak{a} \subset \mathfrak{g}^{-1,1} \text{ is abelian}\right\}.$$

Then

$$\begin{cases} L(\mathfrak{g}) = 0 \iff D \text{ is a Hermitian symmetric domain} \\ L(\mathfrak{g}) > 0 \iff D \text{ is non-classical.} \end{cases}$$

Theorem: *Assuming (GHC) and (GPC), if*

$$\operatorname{tr}\deg(H) < L(\mathfrak{g})$$

the H does **not** *come from geometry.*

- Equivalently,

$$H \text{ comes from geometry} \implies \operatorname{tr}\deg(H) \geqq L(\mathfrak{g}).$$

- For X defined over $\overline{\mathbb{Q}}$ the GPC roughly says that the relations over $\overline{\mathbb{Q}}$ satisfied by the period matrix are reflected in the Mumford-Tate group of the PHS. The extension of the GHC to a general X is due to André is essential for the proof.
- The argument also gives for $H = H^n(X)$ with Mumford-Tate domain D and assuming GPC

$$\operatorname{tr}\deg H < \dim D \implies X \textit{ is } \textbf{not} \textit{ defined over } \overline{\mathbb{Q}}.$$

Example: $n = 2$ and H has Hodge numbers $(2, b, 2)$

$$\operatorname{tr}\deg(H) \leqq b \implies \left\{ \begin{matrix} H \text{ does not come} \\ \text{from geometry} \end{matrix} \right\}.$$

$n = 3$ and H has Hodge numbers $(1, 1, 1, 1)$

$$\operatorname{tr}\deg(H) \leqq 2 \implies \left\{ \begin{matrix} H \text{ does not come} \\ \text{from geometry} \end{matrix} \right\}.$$

References

[BKU] G. Baldi, B. Klingler, and E. Ullmo, On the distribution of the Hodge locus, *Invent. Math.* **235** (2024), no. 2, 441–487.

[R] C. Robles, Schubert varieties as variations of Hodge structure, *Selecta Math. (N.S.)* **20**(3) (2014), 719–768.