Regulators and derivatives of Vologodsky functions with respect to $\log(p)$

Regulators V, Pisa

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K a finite extension of $\mathbb{Q}_p$.

X/K a smooth variety.

Syntomic regulator

$$\mathrm{reg}_{\mathrm{syn}} : H^i_{\mathcal{M}}(X, \mathbb{Q}(j)) \to H^i_{\mathrm{syn}}(X, j)$$

$\mathrm{reg}_{\mathrm{syn}}$ can sometimes be computed using Coleman or Vologodsky integration.

Example: K_2 of a curve X

$\mathrm{reg}_{\mathrm{syn}} : H^2_{\mathcal{M}}(X, \mathbb{Q}(2) \to H^2_{\mathrm{syn}}(X, 2) \cong H^1_{\mathrm{dR}}(X/K) \oplus *$.

$*$ - The star of the talk, was ignored until recently, and for now we keep neglecting it.

To $\omega \in H^1_{\mathrm{dR}}(X/K)$ associate $r_\omega = \omega \cup \mathrm{reg}_{\mathrm{syn}}$.

Theorem (B. 1999)
For $\omega \in \Omega^1(X)$, $r_\omega(\{f,g\}) = \int_{(f)} \log(g)\omega$

Theorem (B. 2021)
In general same formula with Vologodsky integrals (certain restrictions on ω)

What you need to know about p-adic integration:

X/K smooth

Coleman integration (X has good reduction) gives a K-algebra $\mathcal{O}_{\mathrm{Col}}(X)$ of locally analytic functions on X such that

$$0 \to K \to \mathcal{O}_{\mathrm{Col}}(X) \xrightarrow{d} \mathcal{O}_{\mathrm{Col}}(X) \cdot \Omega^1(X)^{d=0} \to 0$$

Vologodsky integration - Same for arbitrary reduction. Depends on the branch of the p-adic logarithm, determined by $\log(p)$.

Coleman integrals depend on the branch only near singular points. The dependency of Vologodsky integrals is more global.

Goal: Show that it is interesting to differentiate a Vologodsky function with respect to $\log(p)$.

Simplest and most important example: The p-adic logarithm:

If ν is the p-adic valuation such that $\nu(p) = 1$, then
$\log(z) = \log(\frac{z}{p^{\nu(z)}}) + \nu(z)\log(p)$ so $\frac{d}{d\log(p)}\log(z) = \nu(z)$.

Apply to the regulator $K^{\times} \to H^1_{\mathrm{st}}(K, \mathbb{Q}_p(1)) = K \oplus \mathbb{Q}_p$

$x \mapsto (\log(x), \nu(x))$

We see the main Theme:

The regulator has 2 components, one "continuous" and one "discrete". The former is computed using Vologodsky integration, The latter is the derivative of the former with respect to $\log(p)$.

p-adic heights

(joint with Müller and Srinivasan)

Line bundle / variety / number field $\mathcal{L}/X/F$

p-adic height $h_{\hat{\mathcal{L}}} : X(F) \to \mathbb{Q}_p$ associated to

$$\hat{\mathcal{L}} = \mathcal{L} + \text{ adelic metric } \|\|_\nu, \quad \nu \text{ finite}$$

Where local metric for $\mathcal{L}/X/K$, K local is

- For p-adic K a *log function*

$$\log : \mathcal{L}^\times : \mathcal{L} - 0 \to \overline{K}$$

  A Vologodsky function satisfying $\log(\alpha w) = \log(\alpha) + \log(w)$, $\alpha \in K^\times$, $w \in \mathcal{L}_x$.
- For $p \neq q$-adic K a *valuation*

$$\text{val} : \mathcal{L}^\times :\to \mathbb{Q}$$

  satisfying $\text{val}(\alpha w) = \nu(\alpha) + \text{val}(w)$ where ν is the valuation on $K^\times$.

Note:

- Valuations can be used for ?-adic heights, including $? = \infty$.
- A good theory of log functions exists (see below).

Main observation: The derivative with respect to $\log(q)$ of a q-adic log function is a q-adic valuation.

Theory of log functions

Theorem (B.)
A log function on $\mathcal{L}/X$ has a curvature $\alpha \in H^1_{\mathrm{dR}}(X/K) \otimes \Omega^1(X)$ cupping to to $ch_1(\mathcal{L})$ (if possible). Conversely, any α cupping to $ch_1(\mathcal{L})$ is the curvature of a log function on $\mathcal{L}$, unique up to an integral of $\omega \in \Omega^1(X)$.

Dynamical situation

$\phi : X \to X$, $\phi^* \mathcal{L} \cong \mathcal{L}^d$.

We can try to fix a "good" norm that makes this an isometry.

For valuations: Limiting techniques (Zhang, like Tate canonical height)

For log functions: Use a curvature α s.t. $\phi^* \alpha = d\alpha$.

Observation: The $\log(q)$ derivative of a good log function is a good valuation.

So, we have two ways of constructing good valuations.

The latter has the following advantage:

If $f : Y \to X$ is a morphism, then f^* val is associated with f^* log, which has curvature $f^*\alpha$, which (in principle) determines it.

Application: Y a curve, X its Jacobian. $f : Y \to X$, $\phi =$ multiplication by 2.

Explicit computation on curves

On curves, curvature $\alpha = \sum [\eta]_i \otimes \omega_i$ means that for any section s

$$\log(s) = \sum \int \left(\omega_i \int \eta_i \right) + \int \gamma$$

where γ is a meromorphic form "that takes care of poles".

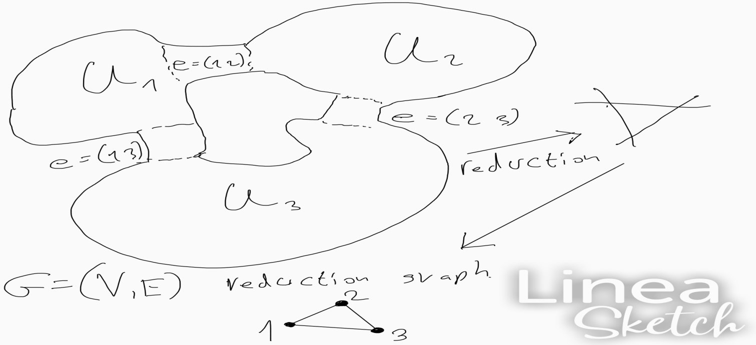
So we need to compute the dependency on $\log(p)$ of $\int (\omega \int \eta)$

This can be done in the semi-stable reduction case using the work of Katz-Litt.

Explicit computation on curves

Consider

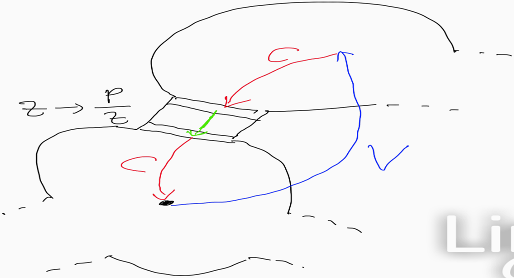
- X/K a complete curve with semi-stable reduction
- $\Gamma = (V, E)$ the reduction graph



Graph convention: Each edge has a start $e+$ and end $e-$ and a reversed edge $-e$ where the order is reversed.

The Katz-Litt Theorem

This recovers Vologodsky integration from Coleman integration.



A loop as above going through an edge e gives rise to a monodromy matrix $A(e)$ (which is always unipotent).

Theorem
The association $e \to A(e)$ is harmonic in the sense that for each $v \in V$, $\sum_{e+=v} \log(A(e)) = 0$.

What does the derivative compute?

Case of unipotent albanese:

X/K a curve.

$G = \pi_1(X, x_0)$ - Fundamental group in the category of filtered (φ, N)-modules.

Albanese map: $\alpha : X(K) \to$ classifying set of G-torsors.

$\alpha(x) =$ space $P_{x_0,x}$ of paths between x_0 and x.

The "continuous part" is computed using Vologodsky integrals:

Vologoksky gives a canonical path $\gamma_{x_0,x,\log(p)} \in P_{x_0,x}$ giving a map

$$x \mapsto \gamma^{-1}_{x_0,x,\log(p)}\gamma \in G/F^0, \quad \gamma \in F^0 P_{x_0,x}$$

which is tautologically given by Vologodsky integrals.

The discrete part

Remember just (φ, N).

Frobenius is trivialized by the Vologodsky path.

Discrete unipotent Albanese into "Space of monodromy operators on G"

The relation with the derivative of the continuous part:

$\gamma_{x_0,x,\log(p)} = I_{\mathrm{HK},\log(p)}(\gamma_{x_0,x})$

$I_{\mathrm{HK},b} = \exp((b-a)N)\, I_{\mathrm{HK},a}$

So deriving in $\log(p)$ picks up N.

Syntomic regulators

We expect a similar picture:

Usually (not cycles) $H^i_{\mathrm{syn}}(X, j) = H^1_{\mathrm{st}}(K, M(j))$

where $M = H^{i-1}_{\text{ét}}(\overline{X}, \mathbb{Q}_p)$

H^1_{st} is semi-stable cohomology

$H^1_{\mathrm{st}}(K, M) = \mathrm{Ext}^1$ in the category of filtered (φ, N)-modules between $(\mathrm{D}_{\mathrm{st}}, \mathrm{DR})$ of $\mathbb{Q}_p$ and M.

Discrete part - forget the filtration.

Continuous part computed with Vologodsky integrals.

$\log(p)$ derivative gives the discrete part.

Toric regulators

Joint work with Wayne Raskind

This is a new type of regulator

It applied to X/K with "totally degenerate" (more or less semi-stable with projective spaces as components) reduction.

Its target is a "p-adic torous" $\mathrm{CoKer} : T^0 \to T^{-1} \otimes K^\times$, T^* finitely generated $\mathbb{Z}$-modules.

Some examples of toric regulators

- Identity map $H^1_{\mathcal{M}}(K,1) = K^\times \to K^\times$
- For E/K a Tate elliptic curve $\mathbb{G}_m/q^{\mathbb{Z}}$ the identity map

$$H^2_{\mathcal{M}}(E,1)_0 \cong E(K) \to K^\times/q^{\mathbb{Z}}$$

- More generally, for a Mumford curve X with Jacobian J, the Abel map $X(K) \to J(K)$, $J(K)$ given via its p-adic uniformization.
- The Pal regulator: For X a Mumford curve with dual graph Γ

$$K_2(X) \to \mathcal{H}(\Gamma, K^\times) = \text{ Harmonic cochains}$$

Construction of the toric regulator

Special fiber Y decomposes as a union $Y = \cup_{i=1}^{n} Y_i$. s.t., with

$$Y_I = \bigcap_{i \in I} Y_i, I \subset \{1, \ldots, n\}$$

$k, r \geq 0$

$M_\ell = H^k_{\text{ét}}(X \otimes_K \bar{K}, \mathbb{Z}_\ell)$

Theorem (Raskind and Xarles)
Ignoring finite torsion and cotorsion there are finitely generated $\mathbb{Z}$-moduels T^i, $i \in \mathbb{Z}$ such that for each ℓ the Galois module $M_\ell(r)$ have an increasing filtration U with $\mathrm{gr}_U M_\ell(r) = \oplus_i T^i \otimes \mathbb{Z}_\ell(-i)$.

Proof: Rapoport-Zink weight spectral sequence for $\ell \neq p$

Mokrane + Hyodo + Tsuji when $\ell = p$.

Assume trivial action of $\mathrm{Gal}(\bar{K}/K)$ on the T's.

Construction of T's

$\bar{Y}_I := Y_I \otimes \bar{F}$

$\bar{Y}^{(m)} = \bigcup_{|I|=m} \bar{Y}_I$

$C_j^{i,k} = CH^{i+j-k}(\bar{Y}^{(2k-i+1)}),\ k \geq \max(0,i)$

$C_j^i = \bigoplus_k C_j^{i,k}$

$I_r = I - \{i_r\}$

Inclusion $\rho_r : Y_I \to Y_{I_r}$

$$\theta_{i,m} = \sum_{r=1}^{m+1} (-1)^{r-1} \rho_r^* : CH^i(\bar{Y}^{(m)}) \to CH^i(\bar{Y}^{(m+1)}) \ ,$$

$$\delta_{i,m} = \sum_{r=1}^{m+1} (-1)^r \rho_{r*} : CH^i(\bar{Y}^{(m+1)}) \to CH^{i+1}(\bar{Y}^{(m)}) \ ,$$

$$d' = \bigoplus_{k \geq \max(0,i)} \theta_{i+j-k, 2k-i+1} \ ,$$

$$d'' = \bigoplus_{k > \max(0,i)} \theta_{i+j-k, 2k-i} \ ,$$

$d_j^i = d' + d'' : C_j^i \to C_j^{i+1}$

$T_j^i := H^i(C_j^\bullet)$

Renumbering: $T^i = T_{i+r}^{k-2r-2i}$

Monodromy map: $N : T^i \to T^{i-1}$ given by "identity on identical components".

The toric intermediate Jacobian

$M'_\ell = U^0 M_\ell(r)/U^{-2} M_\ell(r)$

$0 \to T^{-1} \otimes \mathbb{Z}_\ell(1) \to M'_\ell \to T^0 \otimes \mathbb{Z}_\ell \to 0$

Use Bloch-Kato H_g. Boundary map

$\tilde{N}_\ell : T^0 \otimes \mathbb{Z}_\ell \to H^1_g(K, T^{-1}) \otimes \mathbb{Z}_\ell(1)) \cong T^{-1} \otimes K^{\times(\ell)}$

Lemma

$T^0 \otimes \mathbb{Z}_\ell \xrightarrow{\tilde{N}_\ell} T^{-1} \otimes K^{\times(\ell)} \xrightarrow{\text{val}} T^{-1} \otimes \mathbb{Z}_\ell$ *is* $N \otimes \mathbb{Z}_\ell$.

Corollary

Exists $T^0 \xrightarrow{\tilde{N}} T^{-1} \otimes K^\times$ *s.t.* $\tilde{N}_\ell = \tilde{N} \otimes \mathbb{Z}_\ell$ *for each* ℓ

We define the toric intermediate Jacobian

$$H^{k+1}_{\mathcal{T}}(X, \mathbb{Z}(r)) := \text{CoKer } \tilde{N}$$

Toric regulator completed at ℓ

The etale regulator map $\mathrm{reg}_\ell : H^{k+1}_{\mathcal{M}}(X, \mathbb{Z}(r))_0 \to H^1_g(K, M_\ell(r))$ factors via $H^1_g(K, U^0 M_\ell(r))$ because $H^0(K, \mathbb{Q}_\ell(j)) = H^1_g(K, \mathbb{Q}_\ell(j)) = 0$ for $j < 0$.

Applying $U^0 \to M'_\ell$ we get

$$\mathrm{reg}''_\ell : H^{k+1}_{\mathcal{M}}(X, \mathbb{Z}(r))_0 \to H^1_g(K, M'_\ell) \cong \mathrm{CoKer}(T^0 \otimes \mathbb{Z}_\ell \xrightarrow{\tilde{N}_\ell} T^{-1} \otimes K^{\times(\ell)})$$

The Sreekantan regulator

Targets in "Deligne style cohomology". In most cases this is

$H_{\mathcal{D}}^{k+1}(X, \mathbb{Q}(r)) \cong \mathrm{CoKer}(T_{\mathbb{Q}}^{0} \to T_{\mathbb{Q}}^{-1})$

Assuming standard conjectures Sreekantan's Deligne cohomology is isomorphic to higher Chow groups of the special fiber

$$H_{\mathcal{D}}^{k+1}(X, \mathbb{Q}(r)) \cong CH^{r-1}(Y, 2r-k-2) \otimes \mathbb{Q}$$

and the regulator is just a boundary map in K-theory.

Conjecture
For each prime ℓ the valuation of the toric regulator at ℓ is the Sreekantan regulator tensored with $\mathbb{Q}_\ell$.

Theorem
Assuming Conjecture, Exists $H_{\mathcal{M}}^{k+1}(X, \mathbb{Z}(r)) \xrightarrow{\mathrm{reg}_t} H_{\mathcal{T}}^{k+1}(X, \mathbb{Z}(r))$ giving the toric regulator at ℓ for each ℓ by completion.

Relation with the log-syntomic regulator

Moto: The log-syntomic regulator is the logarithm of the toric regulator.

Natural since: "The syntomic Abel-Jacobi map is the log composed with Albanese"

$\mathrm{reg}_{\mathrm{syn}} : H^{k+1}_{\mathcal{M}}(X, \mathbb{Z}(r))_0 \to H^1_{\mathrm{st}}(K, V)$, $V = H^k_{\text{ét}}(X \otimes_K \bar{K}, \mathbb{Q}_p(r))$

Commuting diagram

$$\begin{array}{ccccc} H^{k+1}_{\mathcal{M}}(X, \mathbb{Z}(r))_0 & \longrightarrow & H^1_{\mathrm{st}}(K, V) & \longrightarrow & \mathrm{DR}(V)/(F^0 + T^0 \otimes \mathbb{Q}_p) \\ \downarrow & & & & \downarrow \\ H^{k+1}_{\mathcal{T}}(X, \mathbb{Z}(r)) & & \xrightarrow{\quad \log \quad} & & T^{-1} \otimes K / T^0 \otimes \mathbb{Q}_p. \end{array}$$

Expectation: The derivative with respect to $\log(p)$ of the syntomic regulator gives the Sreekantan regulator.

This can be checked for example on K_2 of Mumford curves.

X a Mumford curve with dual graph $\Gamma = (V, E)$

Y_v - component of reduction

The Sreekantan regulator

$$K_2(X) \to (v \mapsto k(Y_v)^\times) \to \mathcal{H}(\Gamma, \mathbb{Z})$$

$$\{f, g\} \mapsto (v \mapsto h_v = t_{Y_v}(f, g)) \mapsto (e \mapsto \operatorname{ord}_e(h_{e+}) - \operatorname{ord}_e(h_{e-}))$$

The syntomic regulator

$$K_2(X) \to \mathcal{H}(\Gamma, K), \quad \{f, g\} \mapsto \operatorname{res}_e \log(f) d\log(g)$$

Suppose $\operatorname{ord}_{Y_v}(f, g) = (1, 0)$. Then $\log(f) d\log(g) = \log(p) d\log(g) + \dots$ so

$$\frac{d}{d\log(p)} \operatorname{res}_e \log(f) d\log(g) = \operatorname{res}_e d\log(g) = \operatorname{ord}_e(g|_{Y_v})$$